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A method of solution for the creep buckling of structural members of a linear visco-elastic material

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SUMMARY
In this paper a method for solving creep buckling problems for structural members of a linear visco-elastic material is presented. The solution is found in the form of an expansion of the eigenfunctions of the corresponding elastic problem. The method can be used as either an analytic or a numerical technique.

1. Introduction

Many engineering materials such as concrete, polymers, plastics, timber, metals at elevated temperatures etc., exhibit a time dependent behaviour [1]. The behaviour of these materials may often be described, at least over some limited range of interest, by a linear visco-elastic model.

The properties of visco-elastic materials have been extensively studied [1–8] and the solution of many linear quasistatic engineering problems have been obtained by means of the elastic visco-elastic correspondence principle. More recently, numerical methods, making use of finite element techniques, have been developed [9, 10], and have been used to solve more complicated problems.

The problem of the buckling of time dependent materials, creep buckling, has also been extensively studied, see for example, the survey by Hoff [11]. However a great many of these investigations have been related to the creep of metals at elevated temperatures and have therefore involved the use of a non-linear stress strain law; such investigations do not lie within the framework of linear visco-elasticity. The problem of creep buckling of a linear visco-elastic material has been investigated [8, 12–15], but only Lin [12, 13] made use of the correspondence principle and his investigation was restricted to a particularly simple model of visco-elastic behaviour.

It is the purpose of this paper to show how the elastic visco-elastic correspondence principle can be used to derive the solution, from the known elastic solution, for a large class of creep buckling problems.

2. Solution of elastic buckling problems

Many elastic buckling problems are most conveniently thought of as limiting cases of the deformation of an "imperfect" structural member which satisfied an equation of the type:

$$L\{v\}+\lambda(M\{v\}+f)=0\,, \tag{1}$$

where $L\{.\}$, $M\{.\}$ are linear operators, v = deflection of the member, f = imperfection of the system, e.g. initial curvature, eccentricity of applied load etc., $\lambda = Pl^2/EI$, P = applied load, l = a typical length, E = a typical elastic modulus, I = a typical section constant.

In order to simplify the presentation it will be assumed that equation (1) depends on the single independent variable z defined over the interval $0 \leq z \leq l$*. Examples of this type of buckling problem may be found in the standard texts [16, 17].

* The extension to more complicated problems in which there are several independent variables or in which v is a vector is straightforward and examples will be given in Section (5).

The equations to be considered satisfy certain conditions that are generally realized in practice.

(i) It will be assumed that the solution of equation (1) may be written in the form

$$v(z)=\lambda\int_0^l G(\lambda, z, z')f(z')\,dz' \tag{2a}$$

where G is the Green's function for equation (1) and its boundary conditions.

(ii) It will be assumed that equation (1) together with its boundary conditions has a denumerable set of real distinct eigenvalues $\lambda=\lambda_1, \lambda_2, \lambda_3 \ldots$ with corresponding eigenfunctions $v=\phi_1, \phi_2, \phi_3 \ldots$ so that

$$L\{\phi_n\}+\lambda_n M\{\phi_n\}=0\,. \tag{2b}$$

(iii) Finally it will also be assumed that the solution of equation (1) can be expanded in terms of these eigenfunctions, and thus

$$v(z)=\sum_{n=1}^{\infty} v_n\phi_n(z)\,. \tag{2c}$$

Equations (2a, b, c) lead to the conclusion that G is a meromorphic function of λ and can be expanded in the form:

$$G(\lambda, z, z')=\sum_{n=1}^{\infty}\frac{\phi_n(z)\psi_n(z')}{\lambda-\lambda_n}\,. \tag{3a}$$

The functions $\psi_n(z)$ will be in general solutions of the adjoint equation.

Equation (2c) may now be rewritten

$$v(z)=\sum_{n=1}^{\infty}\frac{\lambda f_n\phi_n(z)}{\lambda-\lambda_n} \tag{3b}$$

where

$$f_n=\int_0^l f(z')\psi_n(z')\,dz'\,, \tag{3c}$$

Equation (3b) is of course the usual eigenfunction solution of equation (1).

Notice that equation (3b) implies that, if a load $P=\lambda_n EI/l^2$ is applied to the member under consideration, the deflection v becomes infinite and the member buckles instantaneously. Clearly the buckling load of the member is the lowest such load.

In solving an elastic buckling problem it is not necessary to obtain the complete solution (3b) of equation (1), for since buckling occurs instantaneously, the deflections of the member and the stresses within it only become excessive when the applied load P is close to the critical load $P_c=\lambda EI/l^2$. Thus the ratio $P:P_c$ gives a good indication of the lack of serviceability of the structure. This will not be the case for the buckling of visco-elastic structures for as has been observed by Hoff [11], the consideration of such systems necessitates the adoption of a slightly different view point. This change of view arises because although the deflections of the structure may eventually become large this need not occur in its design life, and so the deflections and stresses of the structure may well not exceed the allowable design deflections and stresses within the design life of the structure. This of course means that, whether or not a particular applied load P will render a structure unserviceable, depends on the design life of the structure, and thus in order to judge whether and when such unserviceability is likely to occur it is necessary to know the deflection of the system for all time $v=v(P, t)$ not merely the critical load P_c.

3. Equations of visco-elasticity

The starting point of many investigations of the stress-strain laws for a visco-elastic material is the observation that the behaviour of a visco-elastic material, in a simple stress state, resembles the behaviour of a rheological model consisting of elastic springs and viscous dashpots joined together by a number of series or parallel connections [6, 8].

For an elastic spring, shown in Fig. 1a, the relation between the stress σ and the strain ε is

$$\sigma = E_0 \varepsilon$$

while for the dashpot shown in Fig. 1b

$$\sigma = \eta_0 \frac{d\varepsilon}{dt} .$$

Both of the above equations are conveniently rewritten

$$\bar{\sigma} = \bar{E}\,\bar{\varepsilon} \tag{4}$$

where the bar notation implies the Laplace transform e.g.

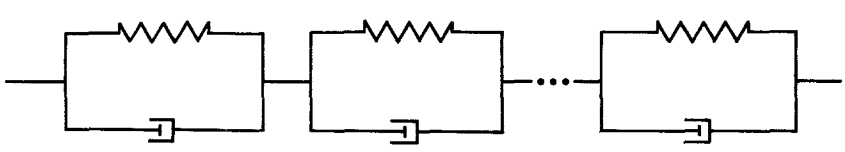
$$\bar{\sigma} = \bar{\sigma}(s) = \int_0^\infty \sigma(t)\, e^{-st}\, dt$$

and where $\bar{E} = \begin{cases} E_0 \text{ for a spring} \\ \eta_0 s \text{ for a dashpot} . \end{cases}$

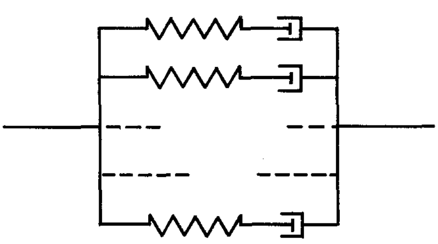
(a) Elastic Spring (b) Viscous Dashpot

Figure 1.

$\bar{E}$ is called the transformed modulus of the system. The stress-strain law for all more complicated models which can be built up by joining springs and dashpots by series and parallel connections can be represented by equation (4) with a suitable definition of $\bar{E}$. It can be seen from equation (4) that for quasistatic loading the equations connecting the Laplace transforms of stress, strain and displacement are identical to those of a corresponding elastic problem. This is the basis of elastic visco-elastic correspondence principle for one dimensional problem. We will be especially interested in the generalized Kelvin model shown in Fig. 2a and the generalized Maxwell model shown in Fig. 2b. These models satisfy equation (4) with



(a) Generalised Kelvin Model



(b) Generalised Maxwell Model

Figure 2.

$$\frac{1}{\bar{E}} = \sum_{i=1}^{p} \frac{1}{E_{ki}+\eta_{ki}s} \quad \text{for a generalized Kelvin model} \tag{5a}$$

$$\bar{E} = \sum_{j=1}^{r} \frac{E_{mj}\eta_{mj}s}{E_{mj}+\eta_{mj}s} \quad \text{for a generalized Maxwell model}\,. \tag{5b}$$

In order that equations (5a, b) should be completely general we must allow the possibility of sub-elements being single springs or dashpots. Thus we must allow the possibility that

$$E_{k1} = 0\,, \quad \eta_{kp} = 0\,, \quad \eta_{m1} = \infty\,, \quad E_{mr} = \infty\,. \tag{5c}$$

Zienkiewicz *et al.* [10] state that many visco-elastic materials can be represented by a generalized Kelvin model. In fact all the models considered in this paper, i.e. those built up by springs and dashpots joined by series and parallel connections, can be represented as both a generalized Kelvin and a generalized Maxwell model. The first step in proving this result is to show the equivalence of generalized Kelvin and Maxwell models. This result follows directly from a lemma due to Bland [6], p. 34, which states that if

$$\Phi = \sum_{r=1}^{m} \frac{\alpha_r}{s+\nu_r} \quad \textit{where} \quad \alpha_r, \nu_r \geqq 0$$

then

$$1/\Phi = E+\eta s + \sum_{r=1}^{m-1} \frac{\beta_r s}{s+\mu_r}$$

where

$$E, \eta, \beta_r, \mu_r \geqq 0\,.$$

Suppose $\Phi = 1/\bar{E}$ as given by equation (5a) then an expansion of the type (5b) is obtained for $\bar{E}$. This shows that a generalized Kelvin model may always be converted to an equivalent generalized Maxwell model. Similarly it may be shown that a generalized Maxwell model can always be converted to an equivalent generalized Kelvin model.

The final step of the proof is most easily obtained by induction. Consider a system S consisting of p elements and suppose that the theorem is true for all values of m, $m<p$. It may be assumed that $p>1$ for otherwise the theorem is trivially true. Since the elements of the model are joined by series or parallel connections it is always possible to subdivide the system into sub-systems S_1, S_2 containing p_1, p_2 elements respectively where $1 \leqq p_1 < p$, $1 \leqq p_2 < p$, $p = p_1+p_2$. Clearly both p_1, p_2 are less than p and thus the theorem may be taken as true for the systems in S_1, S_2. There are now just two possibilities

(i) S_1, S_2 are connected in series. In this case represent S_1, S_2 by generalized Kelvin model. Then S is a generalized Kelvin model. It follows from Bland's lemma that S can also be represented by a generalized Maxwell model.

(ii) S_1, S_2 are connected in parallel. In this case represent S_1, S_2 by generalized Maxwell models. Then S is a generalized Maxwell model which by Bland's lemma can also be represented by a generalized Kelvin model.

Thus it has been shown that if the theorem is true for all $m<p$ then it is true for $m=p$. The result is clearly true when $p=1, 2$ and so the theorem follows by induction.

The extension of the concepts of linear visco-elasticity to three-dimensional isotropic situations were pioneered by Lee [4] and Bland [6]. This extension relies on the observation that, if a material is isotropic, the application of a hydrostatic stress leads only to a dilation of the material. This observation when coupled with the assumption of isotropy implies that a deviatoric stress produces only a deviatoric strain. For such a material the deviatoric and volumetric behaviour are uncoupled. It is then usual to suppose that the volumetric stress and strain are related by one mechanical model while the deviatoric stresses and strains are related by another. Thus if σ_{ij}, ε_{ij} denote the cartesian components of the stress and strain tensors and s_{ij}, e_{ij} denote the components of the stress and strain deviators

$$s_{ij} = \sigma_{ij} - \tfrac{1}{3}\sigma_{kk}\delta_{ij}\,, \quad e_{ij} = \varepsilon_{ij} - \tfrac{1}{3}\varepsilon_{kk}\delta_{ij}$$

then the stress strain law may be written in the form

$$\bar{s}_{ij} = 2\bar{G}\bar{e}_{ij}\,, \quad \bar{\sigma}_{kk} = 3\bar{K}\bar{\varepsilon}_{kk} \tag{6a, b}$$

$\bar{G}$, $\bar{K}$ are derived from mechanical models of the type discussed earlier in this section. These models can be represented as a generalized Kelvin model

$$\frac{1}{\bar{G}} = \sum_{i=1}^{m_d} \frac{1}{E_{di}+\eta_{di}s}\,, \quad \frac{1}{\bar{K}} = \sum_{i=1}^{m_v} \frac{1}{E_{vi}+\eta_{vi}s} \tag{7a, b}$$

and where the limiting cases of the type described by equation (5c) are still possible. Again it can be seen that for quasistatic problems the equations connecting the Laplace transforms of stress, strain and displacement are formally identical to those of a corresponding elastic problem and thus equations (6a, b) provide the basis of the correspondence principle for quasi-static problems as detailed in References [3, 4].

It is sometimes useful to rewrite equations (6a, b) in the form

$$\bar{\varepsilon}_{ij} = \frac{1+\bar{\nu}}{\bar{E}}\,\bar{\sigma}_{ij} - \frac{\bar{\nu}}{\bar{E}}\,\bar{\sigma}_{kk}\delta_{ij} \tag{8a}$$

where

$$\bar{\nu} = \frac{\bar{K}-\frac{2}{3}\bar{G}}{2(\bar{K}+\frac{1}{3}\bar{G})}\,, \quad \bar{E} = \frac{3\bar{K}\bar{G}}{(\bar{K}+\frac{1}{3}\bar{G})}\,. \tag{8b, c}$$

In order that the theory should be self consistent it must be shown that the equation (8c) derived from three-dimensional theory does not conflict with the one-dimensional theory, equations (5a, b, c). That is that the axial behaviour can be derived from a generalized Kelvin or generalized Maxwell model. To do this it is useful to consider one dimensional models such that

$$\bar{\sigma} = \bar{G}\bar{\varepsilon} \quad \text{(Deviatoric model)}$$

$$\bar{\sigma} = \bar{K}\bar{\varepsilon} \quad \text{(Volumetric model)}\,,$$

Equation (8c) can be rewritten in the form

$$\frac{1}{\bar{E}} = \frac{1}{3\bar{G}} + \frac{1}{9\bar{K}}\,.$$

When this equation is interpreted in terms of mechanical models, the axial model shown in Fig. 3 is obtained. This is clearly of the type considered in the one-dimensional theory and thus

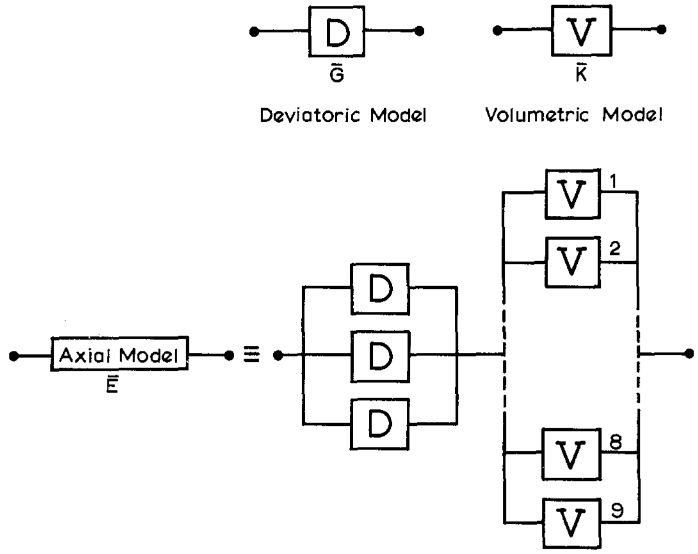


Figure 3.

the axial model derived from the three-dimensional theory is consistent with the one of one-dimensional theory.

It will be useful in later applications to consider a visco-elastic material which has a constant Poisson's ratio, this may prove a close approximation (over a limited range) for many materials and has the advantage that it greatly extends the class of problems for which a solution can be found. For such a problem there is just one rheological model as the volumetric and deviatoric models are simply proportional.

4. Solution of visco-elastic buckling problems

In many problems it is found that the deflection, of an "imperfect" visco-elastic member which obeys stress strain laws of the type (6, 7), is governed by an equation of the form:

$$L\{\bar{v}\}+\bar{\lambda}(M\{\bar{v}\}+\bar{f})=0 \tag{9a}$$

where the notation of Section 2 has been retained and where $\bar{\lambda}=Pl^2/\bar{E}I$, $\bar{E}$=the transformed elastic modulus, $\bar{f}=f/s$ describes the initial imperfections of the system.

In an elastic analysis the imperfection f does not play an important role since the structure will buckle instantaneously at the critical load $P=P_c$ no matter what form this imperfection takes, although for loads less than the critical load the imperfection controls the limit of linear behaviour which governs the range of validity of the theory. However as was indicated previously, in designing a visco-elastic structure it is important to have some idea of the way in which the deflection varies with time and thus for visco-elastic problems the imperfection may play a non-trivial role. As for the elastic case the imperfection will control the limit of linear behaviour and thus the range of validity of the theory.

It will be assumed that the boundary conditions of the visco-elastic problem, equation (9a), are identical to the boundary conditions of the analogous elastic problems, equation (1). This means that the eigenfunctions $\phi_n(z)$ of equation (1) automatically satisfy the boundary conditions of equation (9a). In fact equation (9a) and equation (1) are identical except for the change of notation:

$$v\to\bar{v}\,,\quad \lambda\to\bar{\lambda}\,,\quad f\to\bar{f}\,.$$

The solution of equation (9a) therefore follows directly from equations (3) and thus

$$\bar{v}=\sum_{n=1}^{\infty}\frac{f_n\bar{\lambda}}{s(\lambda_n-\bar{\lambda})}\,\phi_n(z)\,. \tag{9b}$$

It will be assumed that the elastic solution, equation (3b), is known *a priori*. Thus there is no need to enter into a discussion of the validity of the operations leading to equation (9b) as it may be assumed that these will have been justified in establishing the elastic solution.

The final step in the solution is to obtain the inverse Laplace transform of equation (9b). To do this it is convenient to introduce the function $A(t,\alpha)$ defined by means of its Laplace transform

$$\bar{A}=\frac{E_\infty/\bar{E}}{s(\alpha-E_\infty/\bar{E})} \tag{10a}$$

where $E_\infty=\lim\limits_{s\to 0}\bar{E}(s)$ (10b)

E_∞ can be identified with the elastic modulus of the material at infinite time, by applying the Tauberian Theorems for Laplace transforms, Reference [18], to equation (4).

Similarly these theorems can be used to show that the quantity E_0 given by

$$E_0=\lim_{s\to\infty}\bar{E}(s)$$

can be identified with the instantaneous elastic modulus of the visco-elastic material.

It was shown previously that the transformed modulus can be derived from a generalized Kelvin model and thus using equation (5a),

$$\frac{E_\infty}{\bar{E}} = \sum_{k=1}^{m} \frac{a_k}{s+b_k} = F(s) \tag{11}$$

where $a_k = E_\infty/\eta_k$, $b_k = E_k/\eta_k$.

No generality is lost if it is assumed that $0 \leqq b_1 < b_2 \ldots < b_m$. The function $F(s)$ defined by equation (11) has the following easily derived properties:

(i) $F(s) \to +\infty$, $s \to -b_k + 0$
$F(s) \to -\infty$, $s \to -b_k - 0$
(ii) $F'(s) < 0$ for all $s \neq -b_k$
(iii) $F(s) \to E_\infty/E_0$ as $s \to \infty$
E_0 denotes the elastic modulus for $t = 0+$. (Note that $E_0 \to \infty$ if $\eta_k \neq 0$, $k = 1, \ldots, m$; however if $\eta_m = 0$ then E_0 takes a finite value.)
(iv) $F(s) = 1$ when $s = 0$.

Properties (i, ii) follow from the definition of $F(s)$ while properties (iii, iv) follow from the Tauberian theorems for Laplace transforms. These properties show that $F(s)$ can be represented schematically as shown in Fig. 4.

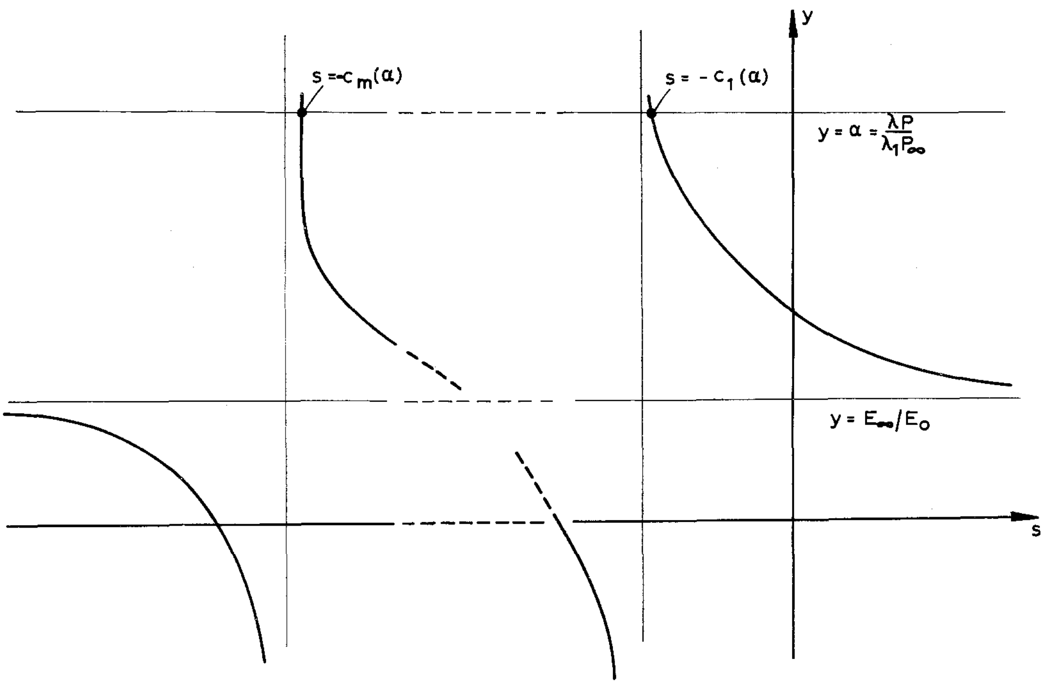


Figure 4. Schematic representation of the function $y = F(s)$.

The line $y = \alpha$ intersects the curve $y = F(s)$ at the roots of $F(s) - \alpha = 0$. Referring to Fig. 4 it is obvious that there are m real roots $s = -c_i(\alpha)$, $i = 1, 2, \ldots, m$. If equation (11) is substituted into equation (10a) it can be seen that $\bar{A}$ is a rational function of s, the degree of the numerator is $m-1$ and the degree of the denominator is $m+1$. If we now expand $\bar{A}$ in partial fractions we see that

$$\bar{A}(s, \alpha) = \frac{A_0(\alpha)}{s} - \sum_{k=1}^{m} \frac{A_k(\alpha)}{s + c_k(\alpha)} \tag{12}$$

where

$$A_0(\alpha) = \frac{1}{\alpha - 1}, \quad A_k(\alpha) = \frac{\alpha}{c_k(\alpha) F'(-c_k(\alpha))}.$$

The inverse Laplace transform of equation (12) is now readily found and thus

$$A(t, \alpha) = A_0(\alpha) + \sum_{k=1}^{m} A_k(\alpha) \exp(-c_k(\alpha)t) . \tag{13}$$

The functions $A(t, \alpha)$ can now be used to obtain the inverse Laplace transform of equation (9b) and thus we have

$$v = \sum_{n=1}^{\infty} f_n A\left(t, \frac{\lambda_n}{\lambda_1} \frac{P_\infty}{P}\right) \phi_n(z) , \tag{14a}$$

where $P_\infty =$ the critical load for a material with elastic modulus $E_\infty = \lambda_1 E_\infty I/l^2$.

Equation (14a) is not in general the most convenient form for numerical calculation. Suppose that v_0 denotes the elastic solution corresponding to an elastic modulus $E=E_0$ then

$$v_0 = \sum_{n=1}^{\infty} f_n A\left(0, \frac{\lambda_n}{\lambda_1} \frac{P_\infty}{P}\right) \phi_n(z)$$

so that

$$v = v_0 + \sum_{n=1}^{\infty} f_n \left[A\left(t, \frac{\lambda_n}{\lambda_1} \frac{P_\infty}{P}\right) - A\left(0, \frac{\lambda_n}{\lambda_1} \frac{P_\infty}{P}\right)\right] \phi_n(z) . \tag{14b}$$

This form of the equation is well suited for calculation at small times.

Similarly if $v=v_\infty$ denotes the elastic solution corresponding to an elastic modulus $E=E_\infty$ then

$$v_\infty = \sum_{n=1}^{\infty} f_n A\left(\infty, \frac{\lambda_n}{\lambda_1} \frac{P_\infty}{P}\right) \phi_n(z)$$

and thus

$$v = v_\infty + \sum_{n=1}^{\infty} f_n \left[A\left(t, \frac{\lambda_n}{\lambda_1} \frac{P_\infty}{P}\right) - A\left(\infty, \frac{\lambda_n}{\lambda_1} \frac{P_\infty}{P}\right)\right] \phi_n(z) . \tag{14c}$$

Equation (14c) is suitable for calculations at large times.

It is possible to discuss the behaviour of equations (14) qualitatively. We shall suppose that $P \geqq 0$ and will denote the critical loads for the elastic problems with $E=E_0$, E_∞ by $P_c = P_0$, P_∞ respectively.

$$P_0 = \lambda_1 E_0 I/l^2 , \quad P_\infty = \lambda_1 E_\infty I/l^2 .$$

First notice that for a given value of P

$$A\left(t, \frac{\lambda_n}{\lambda_1} \frac{P}{P_\infty}\right) > A\left(t, \frac{\lambda_m}{\lambda_1} \frac{P}{P_\infty}\right) > 0$$

when $\lambda_n < \lambda_m$, and thus the behaviour of v will be dominated by the behaviour $A(t, P/P_\infty)$. Now consider the behaviour of the function $A(t, P/P_\infty)$ as P increases from zero. First notice that no matter what the value of $\alpha = P/P_\infty$:

$$c_k(\alpha) > 0 \qquad k = 2, 3, \ldots, m .$$

$c_1(\alpha)$ may be positive or negative, it can be seen from Fig. 3 that

$$c_1(\alpha) > 0 , \quad 0 < P < P_\infty ,$$

$$c_1(\alpha) < 0 , \; P_\infty < P < P_0 ,$$

$$c_1(\alpha) \to \infty , \quad P \to P_0 .$$

It may therefore be concluded that:

(a) If $0 < P < P_\infty$ the functions $A[t, \lambda_n P/(\lambda_1 P_\infty)]$ contain only negative exponentials and hence no buckling occurs.

(b) If $P_\infty < P < P_0$ no instantaneous buckling occurs but because of the presence of positive exponential terms the displacement diverges with time and creep buckling occurs.

(c) If $P=P_0$ instantaneous buckling occurs.
These results are as would be expected and remain valid for the special cases noted in equation (5c).

5. Solution of particular problems

5.1. *Buckling of an axial column*

Consider the eccentrically loaded column shown in Fig. 5. O is the centroid of the section, the column is assumed to have a deflection v in the y direction. It is also assumed that the axial load P is applied at a distance d below the centroidal axis at time $t=0$ and that dynamic

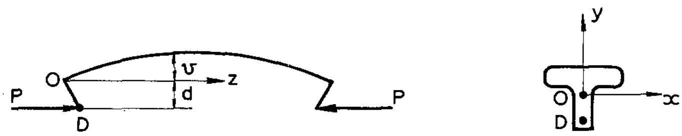


Figure 5.

effects may be neglected. If the usual kinematic assumptions of simple beam theory, are used it is found that

$$\varepsilon = \varepsilon_n + (y-\eta)\frac{\partial^2 v}{\partial z^2} \tag{15}$$

where ε_n = the strain of the neutral axis of bending and $y=\eta$ denotes the position of this axis.

The stress-strain law for the material may be written in the form $\bar{\sigma}=\bar{E}\,\bar{\varepsilon}$ and thus from equation (15)

$$\bar{\sigma} = \bar{E}\left(\bar{\varepsilon}_n + y\frac{\partial^2 \bar{v}}{\partial z^2} - \right)\bar{q} \tag{16}$$

where $\bar{q}$ denotes the Laplace transform of $\eta\,(\partial^2 v/\partial z^2)$. At this stage there is no basis for assuming that the neutral axis of bending coincides with the centroidal axis.

The internal stress σ must be in equilibrium with the axial load P, it then follows from equation (16) that if A is the area of the section

$$P/s = A(\bar{E}\,\bar{\varepsilon}_n - \bar{q})\,. \tag{17a}$$

If it is assumed, as is done in the elastic theory, that ε_n is the axial strain when no bending occurs it is found that

$$P/s = A\bar{E}\,\bar{\varepsilon}_n\,. \tag{17b}$$

Combining equations (17a, b) it follows that $\bar{q}=0$ and thus that $\eta=0$ and so the neutral axis and the centroidal axis coincide.

If now equilibrium of moments is considered, it follows from equation (16) that the moment M is given by

$$\bar{M} = \bar{E}\,I\frac{\partial^2 \bar{v}}{\partial z^2}\,.$$

where I is the second moment of area of the section.

It follows from simple statical considerations that

$$M = -P(v+d) \tag{18}$$

combining the last two equations it is found

$$l^2 \frac{\partial^2 \bar{v}}{\partial z^2} + \frac{Pl^2}{\bar{E}I}\left(\bar{v} + \frac{d}{s}\right) = 0 . \tag{19a}$$

This equation is subject to the boundary conditions

$$\bar{v} = 0 , \quad x = 0 , \tag{19b}$$

$$\bar{v} = 0 , \quad x = l . \tag{19c}$$

Equation (19a) may also be derived as follows. Consider a simple beam subject to moment $M = M(z, t)$, it follows from the theory of elasticity, on suitable approximation, that

$$EI \frac{\partial^2 v}{\partial z^2} = M .$$

The correspondence principle then shows that, provided the approximations made in elastic theory remain valid, viz. equations (15, 17) then for a visco-elastic beam

$$\bar{E}I \frac{\partial^2 \bar{v}}{\partial z^2} = \overline{M}$$

Statical considerations again lead to equation (18) and thus equation (19a) follows.

Equations (19a, b, c) are of the type considered in the previous section. In order to obtain a solution, the eigenvalues and eigenfunctions of the equation,

$$l^2 \frac{\partial^2 v}{\partial z^2} + \lambda v = 0$$

subject to the boundary conditions, $v = 0$ when $x = 0$, $x = l$, are required.

These are well known from the solution to the elastic problem and thus

$$\lambda_n = n^2 \pi^2 , \quad \phi_n = \sin(n\pi z/l) , \quad \psi_n = (2/l) \sin(n\pi z/l) ,$$

and thus for the particular problem being considered

$$f_n = 2d(1 - \cos n\pi)/n\pi$$

and so

$$\bar{v} = \frac{4d}{\pi} \sum_{n=\text{odd}} \frac{\bar{\lambda}}{n^2\pi^2 - \bar{\lambda}} \frac{\cos(n\pi z/l)}{n}$$

whence

$$v = \frac{4d}{\pi} \sum_{n=\text{odd}} \frac{A(t, nP_\infty/P)}{n} \cos(n\pi z/l) .$$

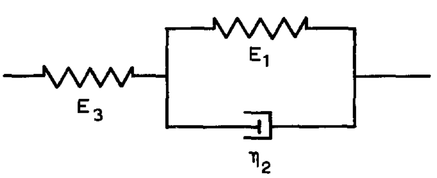


Figure 6.

For the particular case of a material whose behaviour follows the three parameter model shown in Fig. 6 the transformed modulus is given by

$$\frac{1}{\bar{E}} = \frac{1}{E_3} + \frac{1}{E_1 + \eta_2 s} .$$

This model has an elastic response for both small and large time and it is found that

$$E_0 = E_3 , \quad E_\infty = E_1 E_3/(E_1 + E_3) .$$

It follows from the well known elastic theory that

$$P_0 = \pi^2 E_3 I/l^2 \; ; \qquad v_0 = \frac{d(\cos[\pi(P/P_0)^{\frac{1}{2}}(z/l-\frac{1}{2})]-1)}{\cos[\pi(P/P_0)^{\frac{1}{2}}/2]}$$

and

$$P_\infty = \frac{\pi^2 E_1 E_3 I}{(E_1+E_3)l^2}, \quad v_\infty = \frac{d(\cos[\pi(P/P_\infty)^{\frac{1}{2}}(z/l-\frac{1}{2})]-1)}{\cos[\pi(P/P_\infty)^{\frac{1}{2}}/2]}$$

for this particular material

$$\bar{A}\left(s, \frac{n^2 P_\infty}{P}\right) = \frac{P}{n^2 P_\infty - P}\left[\frac{1}{s} - \frac{n^2(P_0-P_\infty)}{(n^2 P_0 - P)(s+\gamma_n/\tau)}\right]$$

where

$$\tau = \eta_2/E_1\,, \quad \gamma_n = \frac{P_0(n^2 P_\infty - P)}{P_\infty(n^2 P_0 - P)}$$

and thus

$$\frac{A(t, n^2 P_\infty)}{P} = \frac{P}{n^2 P_\infty - P}\left[1 - \frac{n^2(P_0-P_\infty)}{(n^2 P_0 - P)}\, e^{-\gamma_n t/\tau}\right]$$

either of the forms

$$v = v_0 + \sum_{n=\text{odd}} \frac{n(P_0-P_\infty)P}{(n^2 P_\infty - P)(n^2 P_0 - P)}(1-e^{-\gamma_n t/\tau}) \sin\frac{(n\pi z)}{l}$$

$$v = v_\infty + \sum_{n=\text{odd}} \frac{n(P_0-P_\infty)P}{(n^2 P_\infty - P)(n^2 P_0 - P)}\, e^{-\gamma_n t/\tau} \sin\frac{(n\pi z)}{l}$$

may be used for numerical calculations.

5.2. *Buckling of a plate*

Suppose O_x, O_y, O_z are a set of reference axes, consider a plate lying in the plane $z=0$, let w be the deflection of the plate in the z-direction. If this plate is acted upon by a resultant distributed load $Q(x, y, t)$ in the z-direction then it follows, from the theory of elasticity together with the assumptions of small deflections of thin elastic plates, that

$$D\left(\frac{\partial^4 w}{\partial x^4} + \frac{2\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) = Q$$

where $D = Eh^3/12(1-v^2)$, h being the plate thickness.

It now follows from the correspondence principle that if the same approximations remain valid for a visco-elastic plate then

$$\bar{D}\left(\frac{\partial^4 \bar{w}}{\partial x^4} + \frac{2\partial^4 \bar{w}}{\partial x^2 \partial y^2} + \frac{\partial^4 \bar{w}}{\partial y^4}\right) = \bar{Q} \tag{20}$$

where $\bar{D} = \bar{E}h^3/12(1-\bar{v}^2)$.

Suppose that the plate is subject to an actual constant applied load q together with in-plane forces N_x, N_y, N_{xy}. If the deformed shape of the plate is taken into account, these in plane forces will contribute a component in the vertical direction and it may be shown ([16]) that the resultant distributed load is

$$Q = q + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y}\,.$$

Equation (20) then becomes

$$\bar{D}\left(\frac{\partial^4 w}{\partial x^4} + \frac{2\partial^4 \bar{w}}{\partial x^2 \partial y^2} + \frac{\partial^4 \bar{w}}{\partial y^4}\right) = \frac{q}{s} + N_x \frac{\partial^2 \bar{w}}{\partial x^2} + 2N_{xy}\frac{\partial^2 \bar{w}}{\partial x \partial y} + N_y \frac{\partial^2 \bar{w}}{\partial y^2}\,. \tag{21}$$

Consider the simply supported rectangular plate, shown in Fig. 7 which is subject to an axial thrust $N_x = -N$. For this case it is necessary to solve equation

$$a^2 b^2 \left(\frac{\partial^4 \bar{w}}{\partial x^4} + 2 \frac{\partial^4 \bar{w}}{\partial x^2 \partial y^2} + \frac{\partial^4 \bar{w}}{\partial y^4} \right) + \bar{\lambda} \left(\frac{\partial^2 \bar{w}}{\partial x^2} - \frac{q(x, y)}{N \cdot s} \right) = 0$$

subject to the boundary conditions

$$\bar{w} = 0, \; \nu \frac{\partial^2 \bar{w}}{\partial y^2} + \frac{\partial^2 \bar{w}}{\partial x^2} = 0 \,, \quad x = 0 \,, \; x = a \,,$$

$$\bar{w} = 0, \; \nu \frac{\partial^2 \bar{w}}{\partial x^2} + \frac{\partial^2 \bar{w}}{\partial y^2} = 0 \,, \quad y = 0 \,, \; y = b \,,$$

where it has been assumed that the material has a constant Poisson's ratio, i.e. $\bar{\nu} = \nu$, and where $\bar{\lambda} = Na^2 b^2 / \bar{D}$.

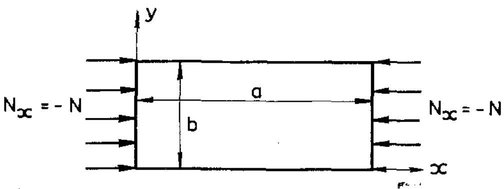


Figure 7.

The elastic eigenvalues are

$$\lambda = \lambda_{mn} = \frac{(m^2 \pi^2/a^2 + n^2 \pi^2/b^2)^2}{m^2 \pi^2/a^2} a^2 b^2 \,, \qquad m = 1, 2, 3 \ldots, \; n = 1, 2, 3 \ldots$$

and the corresponding eigenfunctions

$$w = \phi_{mn} = \sin(m\pi x/a) \sin(n\pi y/b)$$

and so it is found that:

$$\bar{w} = -\frac{1}{N} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\bar{\lambda}}{s(\lambda_{mn} - \bar{\lambda})} q_{mn} \sin(m\pi x/a) \sin(n\pi y/b) \tag{22}$$

where

$$q_{mn} = \frac{4a}{Nm^2 \pi^2 b} \int_0^a \int_0^b q \sin(m\pi x/a) \sin(n\pi y/b) \, dx \, dy$$

and thus

$$w = -\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} q_{mn} A \left(t, \left(\frac{b^2 m^2 + a^2 n^2}{b^2 + a^2} \right)^2 \frac{N_\infty}{m^2 N} \right) \sin(m\pi x/a) \cdot \sin(n\pi y/b) \,. \tag{23}$$

Where N_∞ denotes the critical load for an elastic plate with $E = E_\infty$,

$$N_\infty = D_\infty (\pi^2/a^2 + \pi^2/b^2)^2 / (\pi^2/a^2) \,.$$

It is perhaps worth noting that in this simple problem the eigenvalues λ_{mn} are independent of Poisson's ratio. Thus equation (22) is valid for the case in which $\bar{\nu} \neq$ constant. An equation of the type (23) can be obtained now using the methods developed in Section 4 provided the mechanical model corresponding to $\bar{D}$ is substituted for the mechanical model corresponding to $\bar{E}$. This model is shown in Fig. 8.

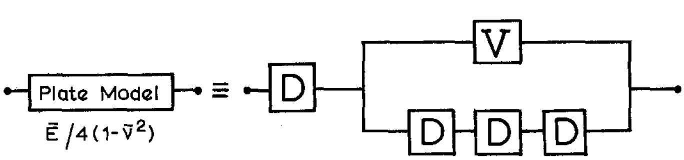


Figure 8.

6. Conclusions

A method of solving creep buckling problems for a linear visco-elastic material has been presented which uses the eigenvalues and eigenfunctions of the simpler elastic problem in order to generate the solution of the visco-elastic problem. The method is extremely flexible and can be applied to a large class of structural creep buckling problems. The range of applicability is considerably extended if it is possible to assume that the Poisson's ratio of material is constant. Several examples of the application of the method have been given. It should be emphasized that the technique is not merely an analytic technique but may be used as a numerical technique, i.e. the eigenvalues λ_n and the eigenfunctions ϕ_n may be determined numerically and need not be known analytically. In practical problems it will generally be sufficiently accurate to consider only the first few eigenvalues as these dominate the subsequent behaviour of the structure.

It should be noted that this theory only remains valid while the material is perfectly visco-elastic. It is no longer applicable when irreversible deformation occurs and plastic regions form. However at this stage the structure is likely to be unserviceable and thus the incidence of first yield forms a convenient basis for estimating the life of the structure.

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